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The Vacuum Pinch in Parallel-Plane Diodes: A Preliminary Examination

by

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The customary electron flow governed by Child's law is laminar and is one-dimensional in plane geometry. In realistic flows edge effects are of more or less importance, depending on the dimensions of the flow and the energy and current. Magnetic forces are fundamentally two-dimensional, and in relativistic, high-current flows, where magnetic forces assume importance comparable with electric ones, the one-dimensional treatment becomes inapplicable. Particularly in short, wide flows the radial electric field becomes short-circuited by the electrodes, and the transverse behavior tends to be dominated by the magnetic field.

Consider the conditions encountered by the outermost ray as the beam radius is increased. The central ray (all rays are for the present purpose taken as injected parallel to the axis) is of course subject to no force and lies along the axis. At small radii outward electric forces dominate, causing some divergence. At radii approaching the electrode spacing d the screening effect of the electrodes begins to be felt so that, due to the magnetic field, the ray again receives little deflection. At still greater radii the magnetic field dominates and ultimately transfers the the initial and acquired forward momentum into inward radial motion.

It is customary to estimate the conditions for which this effect occurs by assuming for the outermost electron the uniform electric and magnetic fields (in Gaussian units) $E = V_o/d$, $H = 2i/r$. We then write

$$d = 3400\ E/(H^2-E^2). \tag{1}$$

This same relation can be written in several different ways. If we set $\gamma_o \equiv 1 + V_o/1700$ and $\beta_o^{\ 2} = 1 - \gamma_o^{\ -2}$, then

$$E/H = \beta_o\gamma_o/(\gamma_o+1). \tag{2}$$

Thus at this limit, and for relativistic beams, E and H for the outside ray are of the same order of magnitude. We can also write

$$i/V_o \cong \tfrac{1}{2}(\gamma_o+1)^{\frac{1}{2}}(\gamma_o-1)^{-\frac{1}{2}}(r_o/d) \tag{3}$$

under the same conditions. This relation can be expressed in practical units (megamperes, megavolts) by the formula

$$i_{MA}/V_{MV} \cong 0.017(1+1/V_{MV})^{\frac{1}{2}}(r_o/d). \tag{3a}$$

The corresponding expression of Child's law (without relativistic corrections) is

$$i_{MA}/V_{MV} = 0.0074\ V_{MV}^{\ \frac{1}{2}}(r_o/d)^2. \tag{4}$$

If these two expressions are set equal, a relation for r_o/d is obtained:

$$r_o/d = 2.24\ V_{MV}^{-\frac{1}{2}}(1+1/V_{MV})^{\frac{1}{2}}. \qquad (5)$$

It can therefore be taken that if r_o/d is below this value, the volt-ampere characteristic of the diode is essentially given by (4) and accordingly depends on $(r_o/d)^2$. In some early thinking (including our own) about the behavior of diodes in which r_o/d exceeds (5) it was assumed that the flow is cut off by magnetron action and that the limit (4) obtains for all greater r_o/d values. A little deeper thought, however, suggests that this conclusion is probably wrong, and that, instead, the volt-ampere behavior for r_o/d greater than (5) is more nearly given by (3a). The flow is thus limited by pinching conditions within the diode region; the resulting crowding of the edge charge into the beam reduces the field near the cathode and thus the current relative to the parallel-plane Child's-law value. The criterion for the determination of the current becomes that condition for which the outermost electron can just cross the gap, supplemented by a new, self-consistent zero-field condition at the cathode. Current increasing linearly with r_o/d is then reasonable (at constant voltage) since this insures that the magnetic field at the beam edge remains the same. This kind of argument raises a question whether the idea of small, closely cycloidal orbits is correct; instead, the flow would appear to be more nearly laminar in view of the requirement $E \sim H$.

In an attempt to determine whether a small-orbit theory can be applied to these devices an equation for guiding-center flow was set up and solved with certain assumed boundary conditions. It is easy to show that at all points the condition $E \gg |a\nabla H|$, where a is the orbit radius, is satisfied; it follows in essence from the fact that for all significant parts of the flow the radius is much greater than the interelectrode spacing. It is therefore supposed that the whole drift is due to the $\underline{E} \times \underline{H}/H^2$ velocity. The equations are

$$\nabla^2 V = 4\pi e n, \tag{6}$$

$$en\beta_g = J(\rho/r)\left[(\partial\rho/\partial r)^2 + (\partial\rho/\partial z)^2\right]^{\frac{1}{2}}, \tag{7}$$

$$\beta_g = (r/2I)\left[(\partial V/\partial r)^2 + (\partial V/\partial z)^2\right]^{\frac{1}{2}}, \tag{8}$$

$$\rho = \text{function of } V \text{ only}, \tag{9}$$

$$I = \int_0^{\rho} 2\pi\rho' \, J(\rho') \, d\rho'. \tag{10}$$

Here J is the density of the axially directed injected current on an injection surface near the cathode (not at the cathode, since with this drift motion the flow lines and equipotentials are the same, and adiabaticity must therefore fail at the electrodes) and ρ is the radius, on the injection surface, at which this current is injected. The scalar current density at any point $en\beta_g$ includes a factor ρ/r for the azimuthal contraction of the injected ring of charge and a factor $(\rho_r{}^2+\rho_z{}^2)^{\frac{1}{2}}$ to account for the inclination of the flow line. Equation (8) is a statement of the small-orbit approximation $\beta_g = E/H$; (9) identifies the flow lines with the equipotentials. The total current injected within ρ is I. Without using (9) we get

$$\nabla^2 V = 4I(dI/d\rho)r^{-2}(\rho_r{}^2+\rho_z{}^2)^{\frac{1}{2}}(V_r{}^2+V_z{}^2)^{-\frac{1}{2}};$$

combining this with (9), we have

$$\nabla^2 V = 4r^{-2}I(dI/d\rho)(d\rho/dV),$$

or simply

$$\nabla^2 V = 2r^{-2}\ dI^2/dV. \qquad (11)$$

The quantity $dI^2/dV \equiv f(V)$, say, is not arbitrary but must be determined so as to meet suitable conditions at the cathode. A method of solution is thus to assume a form of $f(V)$ and then to alter it to improve the match to the required cathode conditions. For present purposes this matching technique will not be required, since substantially different forms of f, namely C and CV, C being a constant, produce similar results in general.

To indicate the general behavior of the solution we first simplify by noting that in all geometries considered $r \gg d$, or $\partial/\partial r \ll \partial/\partial z$, so that we can approximate $\nabla^2 \sim \partial^2/\partial z^2$. Now suppose $dI^2/dV = CV$; define $Z \equiv z/d$, $R \equiv r/d(2C)^{\frac{1}{2}}$, $U \equiv V/V_o$, so that

$$\partial^2 U/\partial Z^2 = U/R^2,$$

and therefore

$$U = (\sinh Z/R)/(\sinh 1/R) \qquad (12)$$

for boundary conditions $U(0,R) = 0$, $U(1,R) = 1$. Also

$$\partial U/\partial Z = R^{-1}(\cosh Z/R)/(\sinh 1/R). \qquad (13)$$

We note that this indeed provides vanishingly small E_z at $z=0$ for all sufficiently small radii. Now continue to suppose, on the small-orbit theory, that $E \ll H$ and define a surface at distance $z_o = 3400\ E/H^2$ from the cathode where the injected orbit is turned back for the first time. Suppose $E = V(z_o)/z_o$; since $I = (\frac{1}{2}C)^{\frac{1}{2}}V$, we have

$$z_o^2 = 1700\ r^2/CV,$$

or, in normalized units

$$(Z_o/R)^2 \sinh Z_o/R = (3400/V_o)\sinh 1/R.$$

The flow is impossible if $Z_o > 1$, and $Z_o < 1$ for $R < (V_o/3400)^{\frac{1}{2}}$. The potential of the guiding center at the extreme value of R is roughly $\frac{1}{2}V_o$. Thus we have $R = r_o/d(2C)^{\frac{1}{2}}$, $i/(\frac{1}{2}V_o) = (\frac{1}{2}C)^{\frac{1}{2}} = \frac{1}{2}r_o/Rd$, $R = (V_o/3400)^{\frac{1}{2}} \sim V_{MV}^{\frac{1}{2}}$ at the outer radius r_o. Combining these, and expressing the result in practical units, we get for the total current

$$i_{MA}/V_{MV} \sim 0.008\ V_{MV}^{-\frac{1}{2}}\ (r_o/d). \qquad (14)$$

This is somewhat similar in form to, though smaller than, (3a). It is nevertheless striking that, having started with a small-orbit approximation, we are led back to the quasi-laminar result in order of magnitude.

The resemblance becomes more pronounced when E/H is calculated with the small-orbit initial assumption. For self-consistency this should be small, at least within the beam. But we have $E \sim \partial V/\partial z = (V_o/d)\partial U/\partial Z$, $H = 2I/r = d(2C)^{\frac{1}{2}}UV_o/rd = (V_o/d)U/R$, so that $E/H = (R/U)\partial U/\partial Z = \coth Z/R$. This is certainly never small compared to 1, so that the resolution violates the initial assumption of small orbits; we are again led back to the $E \sim H$ condition and the corresponding quasi-laminar flow. Qualitatively the initial assumption $dI^2/dV = C$ can be shown to lead to a similar result. We conclude that the orbits are not small, that the flow is quasi-laminar, and that the volt-ampere behavior is approximately (3a). Although these arguments fall short of a true proof, the indication is strongly in favor of this judgment, at least for pure vacuum flow.

Without detailed solution it is not easy to determine accurately the total beam convergence, although a convergence which drives sufficient charge into the beam to reduce the current from (4) to (3a) is evidently required for self-consistency. In the hypothetical absence of magnetic forces we could define a radius of an equivalent parallel beam having the same current as the pinched beam. If r_e is this radius, then

$$(r_e/r_o)^2 = 2.24\, V_{MV}^{-\frac{1}{2}}(1+1/V_{MV})^{\frac{1}{2}}(d/r_o). \qquad (15)$$

The final converged beam radius will be smaller than r_e if the suitably averaged charge density in the equivalent and the pinched beams are to be the same. This argument is incapable of saying how much smaller, but it is probably not violently incorrect to suppose that (15) gives something like the radial convergence of the final beam. This can, however, only be settled by computer analysis.